

# Dual generalizations of sine-Gordon field theory and integrability submanifolds in parameter space

**P. Baseilhac<sup>1,2,3</sup> and D. Reynaud<sup>3</sup>**

<sup>1</sup>*School of Physics, Korea Institute for Advanced Study,  
Seoul, 130-012, Korea*

<sup>2</sup>*Department of Mathematics, University of York,  
Heslington, York YO10 5DD, UK*

<sup>3</sup>*Laboratoire de Physique Mathématique et Théorique  
UMR CNRS 5825, Université Montpellier II  
Place E. Bataillon, 34095 Montpellier, France*

## Abstract

The dual relationship between two  $n - 1$  parameter families of quantum field theories based on extended complex numbers is investigated in two dimensions. The non-local conserved charges approach is used. The lowest rank affine Toda field theories are generated and identified as integrability submanifolds in parameter space. A truncation of the model leads to a conformal field theory in extended complex space. Depending on the projection over usual complex space chosen, a parametrized central charge is calculated.

# 1 Introduction

In statistical mechanics and quantum field theory (QFT), duality plays an important role for exploring strong coupling regime from knowledge of the weak coupling behavior. The sine-Gordon and massive Thirring models are famous examples [1]. In this context, affine Toda field theories (ATFTs) are one-parameter families of quantum integrable massive field theories possessing such duality properties [2].

Recently, we introduced [3, 4] and studied a new  $n-1$  parameter  $(\{\alpha_a\}, \{\beta_a\})$  family of quantum field theories, called multisine-Gordon (MSG) models. The general construction of the MSG in terms of the extended trigonometry associated to the extended complex numbers [5], namely multicomplex (MC) numbers, was obtained starting from a generator  $e$  such that  $e^n = -1$ . For  $(\{\alpha_a\} \in \mathbb{R}, \{\beta_a\} \in \mathbb{R})$ , we have shown that these MSG models provide a unifying representation for a wide variety of integrable QFTs possessing dual representations. In particular, integrable models studied a few years ago [6], like integrable deformations of non-linear  $\sigma$  models or massive Thirring coupled with ATFTs were recovered subject to restrictions on parameter space.

However, a breakthrough comes in [7] from the writing of the extended trigonometric functions (multisine functions), which appeared in the MSG potentials, in terms of the natural multicomplex extension of vertex operators, namely MC-vertex operators. Using this framework, we investigated the existence of non-local conserved charges. In lower-dimensional QFTs, it is well-known that non-local conserved charges may appear, which generate symmetries characterized by braiding relations [8]. They provide a powerful tool for studying non-perturbative effects [9]. A set of equations associated with the conservation of these non-local charges and their algebraic structure was thus obtained.

In this paper, we investigate the dual relationship between two Lagrangian representations generalizing the sine-Gordon model in the previous meaning. Up to restrictions on the parameter space, we show that lowest-rank ATFTs are generated by MC-algebras. Next, description of various integrable perturbations of conformal field theories as different projections over the usual complex space of the same MC-vertex operator is studied.

In section 2, we briefly recall and reinterpret conveniently some results obtained in [3, 4, 7] which constitute the basic ingredient of the following sections. For  $(\{\alpha_a\} \in \mathbb{R}, \{\beta_a\} \in i\mathbb{R})$  (or  $(\{\alpha_a\} \in i\mathbb{R}, \{\beta_a\} \in \mathbb{R})$ ), a generic solution of the equations associated to non-local currents conservation is given. Due to their structure, it naturally emerges a dual relation between the parameters of the model (which appear in the potential through MC-vertex operators) and those involved in the conserved currents. To first order in conformal perturbation theory (CPT), it is then possible to introduce a “dual” family of Lagrangian representations : the “dual” potential is built using the expression in MC-space of conserved currents associated to the original one.

In Section 3, imposing a quantum algebraic structure to the non-local conserved charges, we solve the previous equations. Each solution is associated with a multicomplex space of dimension  $n$ , a specific MC-algebra and different ratios of the parameters. For  $(\{\alpha_a\} \in \mathbb{R}, \{\beta_a\} \in i\mathbb{R})$ , the underlying hidden symmetry of MSG model in each case is identified to a quantum universal enveloping algebra (QUEA) based on an affine Lie algebra  $\hat{\mathcal{G}}$ . This approach provides a unifying “parametrized” description of lowest-rank

QUEAs based on  $A_r^{(1)}$  for  $r \leq 3$ ,  $A_{2r}^{(2)}$  for  $r \leq 2$ ,  $D_4^{(1)}$ ,  $(B_r^{(1)}, A_{2r-1}^{(2)})$  for  $r \leq 3$ ,  $(C_2^{(1)}, D_3^{(2)})$  and  $(G_2^{(1)}, D_4^{(3)})$ .

We show in section 4 how ATFTs are related to MSG models. In particular, simple relations between the multicomplex dimension  $n$  and the rank of  $\hat{\mathcal{G}}$ , the kind of MC-algebra (characterized by  $m_a$ ) and the Kac labels (denoted  $n_a$ ) are obtained. For  $n = 3$  and  $n = 4$ , we describe the (dual-)MC-algebras generating  $A_2^{(1)}$ ,  $C_2^{(1)}$ ,  $D_3^{(2)}$  and  $A_3^{(1)}$  ATFTs.

In section 5, we show that whereas the multisine-potential do not depend on the projections of MC-algebras over the usual complex space, its understanding in terms of perturbed conformal field theory (CFT) does. We identify two kinds of perturbed CFT through the introduction of a (real) multicomplex charge at infinity. For each projection, a “parametrized” central charge is computed.

Some conclusions and perspectives are drawn in section 6.

## 2 Dual conserved currents in extended sine-Gordon

In [3, 4], we introduced the natural extension of the sine-Gordon field theory in the  $n$ -dimensional multicomplex space. This model, generated by the fundamental multicomplex number<sup>1</sup>  $e$  [10, 11], such that  $e^n = -1$ , describes a family of  $n-1$  parameter quantum field theories with  $n-1$  scalar fields which interact through a multisine potential [3, 4]. Its Euclidian action can generally be expressed in terms of the extension in MC-space of standard vertex operators [7] :

$$\mathcal{A}^{(n|m)}(\eta) = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{n\pi} \int d^2z \left( x^{(0)} + \dots + x^{(n-1)} \right) \quad (2.1)$$

with :

$$x^{(l)} = \exp \left( \eta^{(l)} \cdot \Phi(z, \bar{z}) \right), \quad (2.2)$$

where we define :

$$m = 2 \sum_{a=0}^{\frac{n}{2}-1} m_a \quad \text{for } n \text{ even} \quad \text{and} \quad m = 2 \sum_{a=0}^{\frac{n-1}{2}-1} m_a + m_{(n-1)/2} \quad \text{for } n \text{ odd} \quad (2.3)$$

characterize the kind of MC-algebra and  $\Phi(z, \bar{z})$  is the fundamental  $(n-1)$ -components-field of the theory. Note that the index  $l$  denotes the  $l$ -th multicomplex conjugation of any MC-number. In [3], we considered only unimodular MC-numbers. The unimodularity condition was implemented on MC-vertex operators (2.2) through :

$$||x||^n = \prod_{l=0}^{n-1} x^{(l)} = 1, \quad (2.4)$$

where  $|| \cdot ||$  is the pseudo-norm associated to MC-algebra (see refs. [10, 11] for details). Depending on the value of  $n$  (even or odd cases), expression (2.2) differs. For  $n$  even, the

---

<sup>1</sup>More precisely,  $e$  is denoted  $e_{(n|m)}$  in ref. [4] due to its eventual substructure.

MC-vertex operator (2.2) is defined as follow. Firstly, we introduce the bilinear relation [7]:

$$. : \left( \mathbb{MC}_{(n|m)} \right)^{n-1} \times \left( \mathbb{MC}_{(n|m)} \right)^{n-1} \longrightarrow \mathbb{MC}_{(n|m)} \quad (2.5)$$

which can be understood as an extension of the standard scalar product in the multi-complex valued vector space. MC-vertex operators depend on parameters  $(\{\alpha_a\}, \{\beta_a\})$  through the relation :

$$\eta^{(l)} = \left[ \alpha_0 P_l, \dots, \alpha_{\frac{n}{2}-1} P_{\frac{n}{2}-1+l}; \beta_0 Q_{0;l}, \dots, \beta_{\frac{n}{2}-2} Q_{\frac{n}{2}-2;l} \right] \in \left( \mathbb{MC}_{(n|m)} \right)^{n-1}, \quad (2.6)$$

where the parameters  $(\{\alpha_a\}, \{\beta_a\}) \in \mathbb{C}$  and  $\{P_{a+l}, Q_{a;l}\}$  generate the MC-algebra<sup>2</sup> [7]. Consequently, if  $\phi(z)$  denotes the holomorphic part of the fundamental field  $\Phi(z, \bar{z})$ , e.g  $\Phi(z, \bar{z}) = \phi(z) + \overline{\phi(z)}$ , expression (2.2) is defined (since  $\mathbb{C} \subset \mathbb{MC}_{(n|m)}$ ) with :

$$\eta^{(l)} . \phi(z) = \sum_{a=0}^{\frac{n}{2}-1} \left[ \alpha_a P_{a+l} \phi_a(z) \right] + \sum_{a=0}^{\frac{n}{2}-2} \left[ \beta_a Q_{a;l} \varphi_a(z) \right] \quad (2.7)$$

and :

$$Q_{a;l} = -P_{a+l}^2 + \frac{m_a}{m_{n/2-1}} P_{n/2-1+l}^2, \quad Q_{a;l} = Q_{a;l+n/2} = Q_{a+n/2;l} = Q_{a+n;l} = Q_{a;l+n},$$

for  $a \in \{0, \dots, n/2-2\}$ ,  $l \in \{0, \dots, n/2-1\}$  with the conventions  $\alpha_{a+n/2} = \alpha_a$ ,  $\beta_{a+n/2} = \beta_a$  and  $m_{a+n/2} = m_a$ .

In [3, 4], the MSG models were studied for *real* parameters. Let us here focus on the parameter space restricted to :

$$\begin{aligned} & \alpha_a \in \mathbb{R} \quad \text{and} \quad \beta_a \in i\mathbb{R}, \\ \text{or} \quad & \alpha_a \in i\mathbb{R} \quad \text{and} \quad \beta_a \in \mathbb{R}, \end{aligned} \quad (2.8)$$

for all  $a$ . In these cases, we introduce the dual multicomplex element  $\eta^{(k)\vee} = 2\eta^{(k)}(\eta^{(k)}.\eta^{(k)})^{-1}$  (supposing that  $(\eta^{(k)}.\eta^{(k)})$  is invertible which will be always satisfied in the sequel). Using MC-algebra [7] and equation (2.6), it leads to the dual parameter space :

$$\begin{aligned} \alpha_a^\vee &= \frac{-2\alpha_a}{(\alpha_a^2 - \beta_a^2)} \quad \text{for } a \in \{0, \dots, n/2-2\}, \\ \beta_a^\vee &= \frac{-2\beta_a}{(\alpha_a^2 - \beta_a^2)} \quad \text{for } a \in \{0, \dots, n/2-2\}, \\ \alpha_{n/2-1}^\vee &= \frac{-2\alpha_{n/2-1}}{(\alpha_{n/2-1}^2 - \sum_{a=0}^{n/2-2} \frac{m_a^2}{m_{\frac{n}{2}-1}^2} \beta_a^2)}, \end{aligned} \quad (2.9)$$

---

<sup>2</sup>For  $n$  even and “mimimal” representations (i.e  $m_a = 1$  for all  $a$ ), the generators  $P_a$  are expressed [3, 7] in terms of the fundamental multicomplex element  $e$  as :  $P_a = \frac{2}{n} \sum_{j=0}^{n/2-1} \sin[(2a+1)j\frac{\pi}{n}] e^j$ .

and the dual MC-algebra generated by  $e^\vee$  with :

$$\frac{m_a^\vee}{m_{\frac{n}{2}-1}^\vee} = \frac{m_a}{m_{\frac{n}{2}-1}} \frac{(\alpha_a^2 - \beta_a^2)}{(\alpha_{n/2-1}^2 - \sum_{a=0}^{n/2-2} \frac{m_a^2}{m_{\frac{n}{2}-1}^2} \beta_a^2)} \quad \text{for } a \in \{0, \dots, n/2 - 2\} \quad (2.10)$$

In the following, it is then more convenient to rewrite the action of the MSG (2.1) in MC-space as :

$$\mathcal{A}^{(n|m)}(\eta) = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{2\pi} \int d^2z \Phi_{pert}(\eta). \quad (2.11)$$

By analogy, we can now consider the MSG model with action  $\mathcal{A}^{(n|m^\vee)}(\eta^\vee)$ , where  $m^\vee$  is defined as in eq. (2.3) with eqs. (2.10). The perturbing operator for each model reads respectively :

$$\Phi_{pert}(\eta) = \frac{2}{n} \sum_l J_{\eta^{(l)}} \bar{J}_{\eta^{(l)}} \quad \text{and} \quad \Phi_{pert}(\eta^\vee) = \frac{2}{n} \sum_l J_{\eta^{(l)\vee}} \bar{J}_{\eta^{(l)\vee}}. \quad (2.12)$$

Considering now action (2.1) as a perturbed  $(n-1)$  free field CFT and following Zamolodchikov approach [12], it is possible to construct  $2n$  non-local conserved charges to first order in conformal perturbation theory (CPT). Let us consider the holomorphic MC-vertex operator  $J^{(k)} = J_{\eta^{(k)}} = e^{\eta'^{(k)} \cdot \phi(z)}$  (and respectively, for the antiholomorphic part,  $\bar{J}^{(k)} = \bar{J}_{\eta'^{(k)}} = e^{\eta'^{(k)} \cdot \bar{\phi}(z)}$ ). Let us also suppose that its OPE with the  $k^{th}$ -perturbing term of the potential leads to a derivative term :  $\bar{\partial} J^{(k)} = \partial H^{(k)}$  and similarly for the antiholomorphic part (whereas OPEs with any other  $l \neq k^{th}$ -perturbing term yields to regular terms), e.g. the OPE reads :

$$J_{\eta'^{(k)}}(z) x^{(l)}(w) \sim (z-w)^{-C_a^{k,l}(\eta', \eta)} e^{(\eta'^{(k)} + \eta^{(l)}) \cdot \phi(w)} + \dots \quad (2.13)$$

with  $\eta'^{(k)}$  defined as in (2.6) but with  $\alpha_a \rightarrow \alpha'_a$ ,  $\beta_a \rightarrow \beta'_a$  and  $m_a \rightarrow m'_a$ . Since MC-exponents  $C_a^{k,l}(\eta', \eta) = \eta'^{(k)} \cdot \eta^{(l)}$  (reported in [7]) are of the form  $C_a^{k,l}(\eta', \eta) = \sum_{a=0}^{\frac{n}{2}-1} C_a^{k,l}(\eta', \eta) (-P_a^2)$  in terms of MC-algebra, where  $C_a^{k,l}(\eta', \eta)$  depend on  $\alpha_a, \beta_a, \alpha'_a, \beta'_a$ , then using MC-algebra, one easily shows that :

$$J_{\eta'^{(k)}}(z) x^{(l)}(w) \sim \left[ \sum_{a=0}^{\frac{n}{2}-1} (z-w)^{-C_a^{k,l}(\eta', \eta) (-P_a^2)} \right] e^{(\eta'^{(k)} + \eta^{(l)}) \cdot \phi(w)} + \dots \quad (2.14)$$

It clearly implies that the conservation condition of the holomorphic current  $J^{(k)}$  does not depend on the choice of any multicomplex representation. It remains to solve :

$$C_a^{k,k}(\eta', \eta) = 2, \quad C_a^{k,l}(\eta', \eta) \in \mathbb{R}_- \quad \text{for } k \neq l, \quad \text{for all } a, \quad (2.15)$$

with  $(k, l) \in \{0, \dots, n/2 - 1\}$ . Whereas this system of constraints looks overdeterminate, invariance under multicomplex conjugation of action (2.1) leads to redundancies. In

parameter space, the constraints (2.15) reads :

$$\left(-\alpha'_a \alpha_a + \beta'_a \beta_a\right) = 2, \quad \left(-\alpha'_{\frac{n}{2}-1} \alpha_{\frac{n}{2}-1} + \sum_{a=0}^{\frac{n}{2}-2} \beta'_a \beta_a \frac{m'_a m_a}{m'_{\frac{n}{2}-1} m_{\frac{n}{2}-1}}\right) = 2, \quad (2.16)$$

$$\left(\alpha'_a \alpha_a + \beta'_a \beta_a\right) \in \mathbb{R}_-, \quad \left(\alpha'_{\frac{n}{2}-1} \alpha_{\frac{n}{2}-1} + \sum_{a=0}^{\frac{n}{2}-2} \beta'_a \beta_a \frac{m'_a m_a}{m'_{\frac{n}{2}-1} m_{\frac{n}{2}-1}}\right) \in \mathbb{R}_-, \quad (2.17)$$

$$\frac{m_a}{m_{\frac{n}{2}-1}} \beta'_a \beta_a \in \mathbb{R}_+^* \quad \text{and} \quad \frac{m'_a}{m'_{\frac{n}{2}-1}} \beta'_a \beta_a \in \mathbb{R}_+^*, \quad (2.18)$$

for all  $a \in \{0, \dots, n/2 - 2\}$ . If eqs. (2.16), (2.17) and (2.18) are satisfied, the non-local currents  $J^{(k)}$  are conserved to first order in CPT for all  $k$  (and similarly for the antiholomorphic part). They generate  $2n$  non-local conserved charges :

$$\begin{aligned} Q^{(k)} &= \frac{1}{2i\pi} \left( \oint_z dz J^{(k)} + \oint_{\bar{z}} d\bar{z} H^{(k)} \right), \\ \bar{Q}^{(k)} &= \frac{1}{2i\pi} \left( \oint_{\bar{z}} d\bar{z} \bar{J}^{(k)} + \oint_z dz \bar{H}^{(k)} \right) \quad \text{for } k \in [0, \dots, n-1]. \end{aligned} \quad (2.19)$$

The non-locality is due to the fact that (anti-)chiral components  $\phi$  and  $\bar{\phi}$  of the MSG model are non-local with respect to the fundamental field  $\Phi$ . From this property, braiding relations may arise between these charges. It is now obvious to see that the parameters  $\alpha'_a = \alpha_a^\vee$ ,  $\beta'_a = \beta_a^\vee$  and  $m'_a/m'_{n/2-1} = m_a^\vee/m_{n/2-1}^\vee$  satisfy eqs. (2.16), whereas eqs. (2.17), (2.18) still constraint the parameter space  $(\{\alpha_a\}, \{\beta_a\})$  of MSG (2.1). Under these additional conditions on parameter space, conservation of  $J_{\eta^{(k)}} = J_{\eta^{(k)\vee}}$  in the model associated to action  $\mathcal{A}^{(n|m)}(\eta)$  is ensured. It possesses  $2n$  non-local conserved currents  $\{J_{\eta^{(k)\vee}}, \bar{J}_{\eta^{(k)\vee}}\}$ . Consequently, since  $C^{k,l}((\eta^\vee)^\vee, \eta^\vee) = C^{l,k}(\eta^\vee, \eta)$ , the whole set of non-local currents  $\{J_{\eta^{(k)}}, \bar{J}_{\eta^{(k)}}\}$  are similarly conserved to first order in CPT, in the model  $\mathcal{A}^{(n|m^\vee)}(\eta^\vee)$ . It allows to define a duality relation between these two models, at least to order  $\mathcal{O}(\lambda)$ . Note that the transformations (2.9), (2.10) involving  $n-1$  parameters, resulting of the current conservation, is completely independent of any particular structure of non-local conserved charge algebra.

However, to define a consistent QFT, we have to consider carefully the renormalization group flows in such models. The crucial point is that MC-vertex operators  $J_{\eta^{(k)}}$  (and resp.  $\bar{J}_{\eta^{(k)}}$ ) do not form a closed algebra by themselves for general values of parameters. If this condition is not satisfied (which is generally the case), renormalization requires that counterterms have to be added in such a way that this algebra closes. Consequently, the non-local charges are obviously no longer symmetries of this modified action. A well-known example is provided by simply-laced ATFTs with parameter  $\beta$  : for smaller values than the critical value  $\beta^2 = 1$ , only tadpole renormalization is necessary. However, in the Kosterlitz-Thouless region, ATFTs are *not* renormalizable [13] and exponential of *all* the roots<sup>3</sup> must be added in order to render them renormalizable. This situation will be

---

<sup>3</sup>For non-simply laced case, exponential operators associated to short or long roots have drastically different dimensions : one must introduce fermions in such a way as to increase the conformal dimensions of the exponential operators associated to short roots [14].

relevant in further analysis.

### 3 Quantum algebraic structure and lowest-rank affine Lie algebras parametrization

Known integrable models generally exhibit connections with Hopf algebras like Lie algebras and their quantum deformations. The aim of this section is to clarify in which sense the previous parameter space of the multisine-Gordon model is restricted by imposing this kind of structure to the non-local conserved charge algebra. The principal motivation to find such a structure in the MSG models comes from the powerful framework that these algebras provide : they can be sufficiently restrictive to allow a non-perturbative solution of the theory, determining the  $S$  matrices for instance [9]. From eqs. (2.15), i.e. (2.16), (2.17) and (2.18) with solutions (2.9), (2.10), it is convenient to introduce the elements :

$$\begin{aligned}
C_{a,a}(\eta^\vee, \eta) &= 2, & C_{a,a+\frac{n}{2}}(\eta^\vee, \eta) &= C_{a+\frac{n}{2},a} = -2\frac{A_a+B_a}{A_a-B_a} = -n_1^a \\
C_{a,\frac{n}{2}-1}(\eta^\vee, \eta) &= C_{a+\frac{n}{2},n-1} = C_{a+\frac{n}{2},\frac{n}{2}-1} = C_{a,n-1} = 2\frac{B_a\delta_a}{A_a-B_a} = -n_2^a \\
C_{\frac{n}{2}-1,a}(\eta^\vee, \eta) &= C_{n-1,a+\frac{n}{2}} = C_{\frac{n}{2}-1,a+\frac{n}{2}} = C_{n-1,a} = 2\frac{B_a\delta_a}{A_{\frac{n}{2}-1}-B} = -n_3^a \\
C_{n/2-1,n-1}(\eta^\vee, \eta) &= C_{n-1,\frac{n}{2}-1} = -2\frac{A_{\frac{n}{2}-1}+B}{A_{\frac{n}{2}-1}-B} = -n_4
\end{aligned} \tag{3.20}$$

with  $A_a = \alpha_a^2$ ,  $B_a = \beta_a^2$ ,  $\delta_a = \frac{m_a}{m_{\frac{n}{2}-1}}$ , and :

$$B = \sum_{a=0}^{\frac{n}{2}-2} B_a \delta_a^2. \tag{3.21}$$

As detailed in [7], the braiding relations between  $(J^{(k)}, \overline{J}^{(l)})$ ,  $(J^{(k)}, \overline{H}^{(l)})$  and  $(H^{(k)}, \overline{J}^{(l)})$  arising from the non-local property of the currents are identical iff :

$$\{n_1^a, n_4\} \in \mathbb{N} \quad \text{and} \quad \{n_2^a, n_3^a\} \in \mathbb{N}^* \quad \text{for} \quad a \in \{0, 1, \dots, \frac{n}{2} - 2\}. \tag{3.22}$$

Under this assumption, to first order in CPT non-local conserved charges (2.19) obey a  $q$ -deformed structure in MC-space [7]. Furthermore it is possible, in the usual complex space and independently of any representation, to define one other basis of non-local conserved currents; the expansion of the currents  $J^{(k)}$  in MC-basis  $\{P_a, -P_a^2\}$  reads :

$$J_{\eta^{(k)}^\vee} = \sum_{a=0}^{\frac{n}{2}-1} \left[ \sin(\alpha_a^\vee \phi_a) e^{\beta_a^\vee \varphi_a} P_{a+k} + \cos(\alpha_a^\vee \phi_a) e^{\beta_a^\vee \varphi_a} (-P_{a+k}^2) \right] \tag{3.23}$$

with  $\beta_{\frac{n}{2}-1}^\vee \varphi_{\frac{n}{2}-1} = -\sum_{a=0}^{\frac{n}{2}-2} \frac{m_a^\vee}{m_{\frac{n}{2}-1}^\vee} \beta_a^\vee \varphi_a$ . Since  $J_{\eta^{(k)\vee}}$  is conserved, then each one of its components (and any linear combinations of them) is a non-local conserved current, particulary :

$$\begin{aligned}\mathcal{J}^{(a)} &= e^{-i\alpha_a^\vee \phi_a + \beta_a^\vee \varphi_a}, \\ \mathcal{J}^{(a+\frac{n}{2})} &= e^{i\alpha_a^\vee \phi_a + \beta_a^\vee \varphi_a}\end{aligned}\tag{3.24}$$

and similarly for the antiholomorphic part. Non-local conserved charges  $(\mathcal{Q}_a, \overline{\mathcal{Q}}_a)$  associated to these conserved currents can then be obtained as in (2.19). Analogously to the multicomplex case, these charges obey to a  $q$ -deformed algebra iff eqs. (3.20), (3.21) with (3.22) are satisfied. For  $\alpha_a \in \mathbb{R}$  and  $\beta_a \in i\mathbb{R}$ , the resulting structure<sup>4</sup> is nothing else than a “parametrized” quantum universal enveloping algebra  $\mathcal{U}_q(\hat{\mathcal{G}})$  :

$$\mathcal{Q}^{(a)} \overline{\mathcal{Q}}^{(b)} - q_a^{-C_{a,b}(\eta, \eta^\vee)} \overline{\mathcal{Q}}^{(b)} \mathcal{Q}^{(a)} = \delta_{a,b} \frac{\lambda}{in\pi} \left[ 1 - q_a^{2\mathcal{T}^{(a)}} \right].\tag{3.25}$$

where  $q_a = \exp(-i\frac{\pi}{2} C_{a,a}(\eta^\vee, \eta^\vee))$  is the deformation and  $\mathcal{T}^{(a)}$  is a parametrized topological charge [7]. Here, coefficients  $C_{a,b}(\eta, \eta^\vee) = C_{b,a}(\eta^\vee, \eta)$  given by eqs. (3.20) correspond to the extended Cartan matrix elements of this “parametrized”  $q$ -deformed algebra. As we will see later, the type of affine Lie algebra is encoded in the ratios of the parameters and the MC-algebra structure (e.g.  $A_a$ ,  $B_a$  and  $\delta_a$ ). As was shown in [7], the matrix elements of the extended Cartan matrix for  $n$  odd can be obtained similarly from eqs. (3.20) by the substitution :

$$n \rightarrow n+1; \quad A_{\frac{n-1}{2}} \rightarrow 0; \quad \delta_a \rightarrow 2\delta_a,\tag{3.26}$$

while the last equation of (3.20) does not appear. It is more convenient to formulate this substitution in terms of the  $n_i^a$  so that solutions for the  $n$  odd case can be directly read off from solutions of the even case (with  $A_{\frac{n}{2}-1} = 0$ ). It reads :

$$n \rightarrow n+1; \quad n_4 \rightarrow -2; \quad n_2^a \rightarrow \frac{n_2^a}{2}; \quad n_3^a \rightarrow \frac{n_3^a}{2},\tag{3.27}$$

and (3.21) becomes for the odd case :

$$B = 4 \sum_{a=0}^{\frac{n-1}{2}} B_a \delta_a^2.\tag{3.28}$$

In the following, the system (3.20) (and correspondingly for  $n$  odd) is solved<sup>5</sup>. First, using (3.20),  $A_a$ ,  $\delta_a$ ,  $B_a$ ,  $A_{\frac{n}{2}-1}$  are expressed in terms of  $n_i^a$  and  $B$ . Then, equation (3.21), (or (3.28) for  $n$  odd) appear as a further constraint that fix the  $n_i^a$ . Two generic cases are now studied : *i*) no restriction ( $\{\alpha_a\}, \{\beta_a\} \neq 0$  , *ii*) restrictions on parameters space.

<sup>4</sup>For this parameter space, the normalization of the field is chosen such that  $\mathcal{T}^{(a)}$  takes integer values [7].

<sup>5</sup>For completeness, we also report the cases  $C_2^{(1)}$ ,  $D_3^{(2)}$ ,  $A_2^{(1)}$  and  $A_3^{(1)}$  already obtained in [7].

### Case $n$ even without restriction

The case with no restrictions corresponds to keeping  $A_a \neq 0$ ,  $B_a \neq 0$ ,  $A_{\frac{n}{2}-1} \neq 0$ . The general solution for the system (3.20) reads (with  $B < 0$ ) :

$$\begin{aligned} \frac{A_a}{B_a} &= -\frac{2+n_1^a}{2-n_1^a}, & A_{\frac{n}{2}-1} &= -\frac{2+n_4}{2-n_4}B; \\ B_a &= \frac{n_3^a(2-n_1^a)}{n_2^a(2-n_4)}B; & \delta_a &= \frac{2n_2^a}{2-n_1^a}, \end{aligned} \quad (3.29)$$

with  $n_1^a = 0$  or  $1$ ,  $n_2^a > 0$ ,  $n_3^a > 0$  and  $n_4 = 0$  or  $1$ . Since  $n_i^a \in \mathbb{N}^*$ ,  $\delta_a$  are rational and the  $m_a$  can then be integers. Here,  $B$  is a negative real number, and due to (3.21), the  $n_i^a$  must satisfy :

$$\sum_{a=0}^{\frac{n}{2}-2} \frac{n_3^a n_2^a}{2-n_1^a} = \frac{2-n_4}{4}. \quad (3.30)$$

Since  $\frac{n_3^a n_2^a}{2-n_1^a} \in \frac{1}{2}\mathbb{N}^*$ ,  $n_4 = 0$  is the only consistent case. It corresponds to the line  $d/$  of Table 1.1.

### Case $n$ odd without restriction

The solution in this case can be obtained from (3.29) with substitutions (3.27) ( $B < 0$ ):

$$\frac{A_a}{B_a} = -\frac{2+n_1^a}{2-n_1^a}; \quad B_a = \frac{n_3^a(2-n_1^a)}{4n_2^a}B; \quad \delta_a = \frac{n_2^a}{2-n_1^a}, \quad (3.31)$$

with  $n_1^a = 0$  or  $1$ ,  $n_2^a > 0$ ,  $n_3^a > 0$ , and the condition (3.28) :

$$\sum_{a=0}^{\frac{n-1}{2}-1} \frac{n_3^a n_2^a}{2-n_1^a} = 1. \quad (3.32)$$

Bounds on the  $n_i^a$  imply that (3.32) has solutions only if  $n \leq 5$ . There are three solutions for  $n = 3$  corresponding to lines  $a/$ ,  $b/$ ,  $c/$  of Table 1.1, and one for  $n = 5$  (line  $e/$  of Table 1.1).

	$n$	a	$n_1^a$	$n_2^a$	$n_3^a$	$A_a/B_0$	$A_{\frac{n}{2}-1}/B_0$	$B_a/B_0$	$\delta_a$	Algebra
a/	3	0	0	1	2	-1	*	1	1/2	$C_2^{(1)}$
b/	3	0	0	2	1	-1	*	1	1	$D_3^{(2)}$
c/	3	0	1	1	1	-3	*	1	1	$A_2^{(1)}$
d/	4	0	0	1	1	-1	-1	1	1	$(n_4 = 0) A_3^{(1)}$
e/	5	0	0	1	1	-1	*	1	1/2	$D_4^{(1)}$
		1	0	1	1	-1		1	1/2	

Table 1.1 - Solutions without restriction, and the corresponding hidden symmetry algebras.

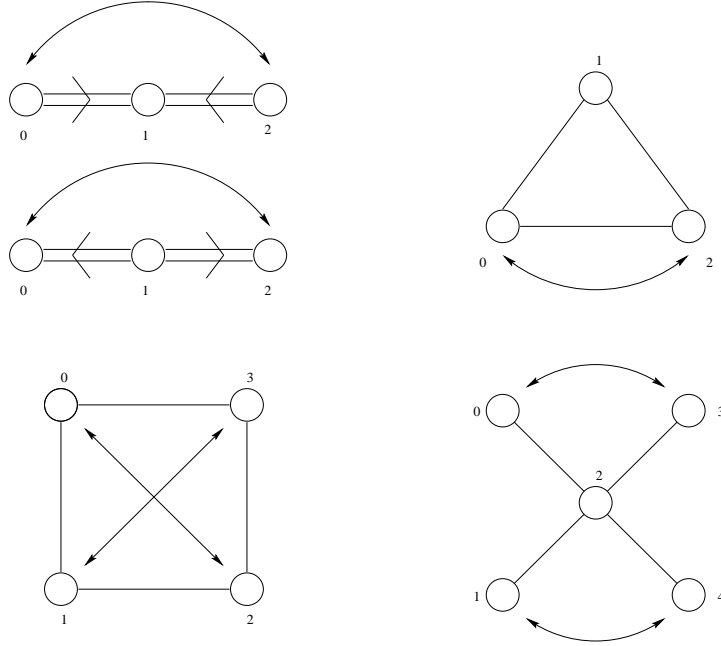


Figure 1.1 - Small arrows symbolise the exchange under multicomplex conjugation  $a \rightarrow a + n/2$  (or  $a \rightarrow a + (n+1)/2$  for  $n$  odd) . The number of links is  $|C_{a,b}(\eta, \eta^\vee)||C_{a,b}(\eta^\vee, \eta)|$  where big arrows goes from  $a$  to  $b$  if  $|C_{a,b}(\eta, \eta^\vee)| > |C_{a,b}(\eta^\vee, \eta)|$ .

### Case $n$ even with restrictions

We consider  $s+1$  restrictions :

$$\begin{aligned} A_a &= 0 \text{ for } a \in \{0, 1, \dots, s\}, \\ A_a &\neq 0 \text{ for } a \in \{s+1, \dots, \frac{n-1}{2} - 1\} \text{ (}\emptyset \text{ if } s = \frac{n-1}{2} - 1\text{)}. \end{aligned} \quad (3.33)$$

The solution of (3.20) for  $a \leq s$  can be obtained from (3.29) by setting  $n_1^a = -2$  :

$$A_a = 0; \quad A_{\frac{n}{2}-1} = -\frac{2+n_4}{2-n_4}B; \quad B_a = \frac{4n_3^a}{n_2^a(2-n_4)}B; \quad \delta_a = \frac{n_2^a}{2}, \quad (3.34)$$

while (3.29) still holds when  $s+1 \leq a \leq \frac{n}{2}-1$ . One can also set the restriction  $A_{\frac{n}{2}-1} = 0$ , in which case solutions of (3.20) are obtained from (3.29) and (3.34) by setting  $n_4 = -2$ , (while  $n_4 = 0$  or  $1$  if  $A_{\frac{n}{2}-1} \neq 0$ ). In all cases, one still have  $n_2^a > 0, n_3^a > 0$ . (3.21) yields the condition :

$$\sum_{a=0}^s \frac{n_2^a n_3^a}{4} + \sum_{a=s+1}^{\frac{n}{2}-2} \frac{n_2^a n_3^a}{2-n_1^a} = \frac{2-n_4}{4}. \quad (3.35)$$

• For  $A_{\frac{n}{2}-1} \neq 0$ , this last equation admits solutions only if  $n \leq 6$  : For  $n = 4$ , there are three solutions written on lines a/, b/, c/ of Table 1.2, and for  $n = 6$  there is one solution line n/ of Table 1.2, reported in Appendix A.

- For  $A_{\frac{n}{2}-1} = 0$ , the upper bound on  $n$  is 10. Solutions are written on lines d/ to m/ and o/ to r/ of Table 1.2 in Appendix A.

### Case $n$ odd with restrictions

In this case, one can only consider  $s + 1$  restrictions :

$$\begin{aligned} A_a &= 0 \text{ for } a \in \{0, 1, \dots, s\}, \\ A_a &\neq 0 \text{ for } a \in \{s+1, \dots, \frac{n-1}{2}\} \ (\emptyset \text{ if } s = \frac{n-1}{2}). \end{aligned} \quad (3.36)$$

Solutions for  $a \leq s$  can be obtained from (3.31) by setting  $n_1^a = -2$  :

$$A_a = 0; \quad B_a = \frac{n_3^a}{n_2^a} B; \quad \delta_a = \frac{n_2^a}{4},$$

while for  $s+1 \leq a \leq \frac{n-1}{2}$  solution is given by (3.31), with  $n_1^a = 0$  or 1. In any case  $n_2^a > 0$  and  $n_3^a > 0$ . The condition (3.28) becomes :

$$\sum_{a=0}^s \frac{n_2^a n_3^a}{4} + \sum_{s+1}^{\frac{n-1}{2}-1} \frac{n_2^a n_3^a}{2 - n_1^a} = 1. \quad (3.37)$$

Owing to the conditions on  $n_i^a$ , and the upper bound of  $s$ , (3.37) have only solutions for  $n \leq 9$ . Solutions are summarized on lines a/ to n/ of Table 1.3 in Appendix A.

To resume, imposing a quantum algebraic structure to the non-local charges restricts the  $(n-1)$ -parameter space to a discrete set of one dimensional submanifolds. Each one is characterized by ratios of the parameters  $(\{\alpha_a\}, \{\beta_a\})$  and ratios of  $m_a$  which determinate  $n_1^a, n_2^a, n_3^a, n_4$  in eqs. (3.20). Consequently, since (3.20) are identified to Cartan matrix elements, any of these submanifolds is in one-to-one correspondance with an affine Lie algebra (see tables. 1.1-1.3). We see that any lowest-rank affine Lie algebra and its dual appear for each fixed value of the multicomplex space of dimension  $n$ . This is easily understood from the substitutions (2.9) and (2.10) in (3.25), resulting from the duality property.

## 4 Integrability in parameter space : affine Toda field theories and perturbed WZNW models

Toda and affine Toda field theories are generally understood as the simple Lie group extension of the Liouville and sine-Gordon models respectively. From our previous results, we intend to show in this section that some of these theories, more precisely those based on lowest rank simply-laced algebras, can be generated starting from a multicomplex number  $e$  satisfying  $e^{10} = -1$ . Let us now consider the set of MC-vertex operators :

$$y^{(l)} = n^{(l)} x^{(l)} \quad \text{with} \quad n^{(l)} = \sum_{a=0}^{\frac{n}{2}-1} n_a (-P_{a+l}^2). \quad (4.38)$$

The pseudo-norm associated to these operators is :

$$||y||^m = ||nx||^m = \prod_{a=0}^{\frac{n}{2}-1} n_a^{2m_a} \quad (4.39)$$

as  $||x||^m = 1$ . It is then straightforward to see that among the models which can be built in terms of these MC-vertex operators, the ones with action (for  $n$  even) :

$$\mathcal{A}^{(n|m)}(\eta) = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{n\pi} \int d^2z \sum_{k=0}^{n-1} n^{(k)} x^{(k)} \quad (4.40)$$

possess exactly the same underlying algebraic structure than those considered in previous sections. Except for the explicit expression of the non-local conserved charges and r.h.s of (3.25), which are modified by the presence of the extra-parameters  $n_a$  appearing through the changes in the non-diagonal terms :

$$H^{(k)} \rightarrow n^{(k)} H^{(k)}, \quad (4.41)$$

in (2.19), the whole analysis concerning the non-local conserved charge algebraic structure is preserved. In the usual complex space (4.40) for  $n$  even writes :

$$\mathcal{A}^{(n|m)}(\eta) = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{2\lambda}{n\pi} \int d^2z \sum_{a=0}^{\frac{n}{2}-1} n_a \cos(\alpha_a \phi_a) e^{\beta_a \varphi_a} \quad (4.42)$$

with  $\beta_{\frac{n}{2}-1} \varphi_{\frac{n}{2}-1} = -\sum_{a=0}^{\frac{n}{2}-2} \frac{m_a}{m_{\frac{n}{2}-1}} \beta_a \varphi_a$ . This action can be generally put into the form :

$$\mathcal{A}^{(n|m)}(\eta) = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{n\pi} \int d^2z \left[ \sum_{a=0}^{n-1} n_a e^{\beta_0 \mathbf{r}_a \cdot \Phi} \right]. \quad (4.43)$$

with  $n_a = n_{a+\frac{n}{2}}$  for  $a \in \{0, \dots, \frac{n}{2} - 1\}$ , and where we introduce some “parametrized” roots  $\mathbf{r}_a$ , reported in Appendix B.

Similarly, to obtain the multisine-Gordon action for  $n$  odd [3, 7], we do successively the substitutions (3.26) in (4.40), (4.42) and (4.43), then change  $n_{\frac{n-1}{2}} \rightarrow n_{\frac{n-1}{2}}/2$ . For  $n$  odd, “parametrized” roots are deduced using the same method. As we see, the form of the action recalls the standard one of affine Toda field theories. Taking the Lagrangian based on the multisine potential, we expand the interaction term around the minimum at  $\Phi = \mathbf{0}$  :

$$\begin{aligned} \sum_{a=0}^{n-1} n_a e^{\beta_0 \mathbf{r}_a \cdot \Phi} &\sim \sum_{a=0}^{n-1} n_a + \beta_0 \sum_{a=0}^{n-1} n_a \mathbf{r}_a^i \Phi^i + \frac{\beta_0^2}{2!} \sum_{a=0}^{n-1} n_a \mathbf{r}_a^i \mathbf{r}_a^j \Phi^i \Phi^j \\ &+ \frac{\beta_0^3}{3!} \sum_{a=0}^{n-1} n_a \mathbf{r}_a^i \mathbf{r}_a^j \mathbf{r}_a^k \Phi^i \Phi^j \Phi^k + \dots \end{aligned} \quad (4.44)$$

Stabilization of the classical vacuum implies cancellation of the linear term. Using (4.42), for each  $n$  even or  $n$  odd (with (3.26)) case this is ensured iff :

$$\frac{n_a}{n_{E(\frac{n+1}{2})-1}} = \frac{m_a}{m_{E(\frac{n+1}{2})-1}} \quad \text{for } a \in \{0, \dots, E(\frac{n+1}{2}) - 2\}, \quad (4.45)$$

where  $E(x)$  stands for the integer part of  $x$ . Consequently, we obtain the linear relation among the “parametrized” roots :

$$\sum_{a=0}^{n-1} m_a \mathbf{r}_a = 0. \quad (4.46)$$

Such kind of relation is characteristic of ATFTs. One of the roots is generally identified with the negative of the highest root of the finite Lie algebra  $\mathcal{G}$  (with rank  $n-1$ ) considered and the set of different numbers  $m_a$  in (4.46) are proportional to the Kac labels. From eq. (4.45), it is interesting to see that ratios of these labels are completely defined by the kind of MC-algebra choosed. Furthermore, using eq. (4.45), the dual Kac labels associated with the dual ATFTs transform as in eq. (2.10). Under these considerations, to each ATFT (and its dual one) corresponds the pseudo-norm (4.39). As the ratios of  $m_a$  take rational values (see the previous section), there exists one faithful representation  $\pi$  [7], given by  $(n \times n)$  dimensional diagonal matrices:

$$\pi[P_a] = \text{Diag}(0, \dots, 0, i, 0, \dots, 0, -i, 0, \dots, 0) \quad \text{for } a \in \{0, \dots, n/2 - 1\}, \quad (4.47)$$

where  $i$  (resp.  $-i$ ) is in the  $a$  (resp.  $n-1-a$ ) position, e.g. :

$$(\pi[P_a])_{jj} = (\pi[P_a^\dagger])_{n-1-j, n-1-j} \quad \text{where } \dagger \text{ denotes the ordinary hermitian conjugate, for } (a, k) \in \{0, \dots, n/2 - 1\} \text{ and } j \in \{0, \dots, n-1\}.$$

For instance, consider the multisine-Gordon model in multicomplex dimension  $n = 3$  denoted by  $MSG_{(3|m)}(\alpha_0, \beta_0)$ , and its dual denoted by  $MSG_{(3|m^\vee)}(\alpha_0^\vee, \beta_0^\vee)$ . From eqs. (2.10) with substitutions (3.26), we have :

$$\frac{m_0^\vee}{m_1^\vee} = \frac{m_1}{m_0} \left[ \frac{1}{2 - n_1^0} \right]. \quad (4.48)$$

As  $m = 2m_0 + m_1$  and similarly  $m^\vee = 2m_0^\vee + m_1^\vee = m + \Delta$  with  $\Delta = m_1 - m_0 n_1^0$ , we obtain three solutions :

- $n_1^0 = 0$  : for  $m_0 = m_1 = 1$  ( $\Delta = 1$ )  $\rightarrow m = n = 3$  we obtain the model denoted  $\mathcal{A}^{(3|3)}(\beta_0, i\beta_0)$  and for  $2m_0^\vee = m_1^\vee = 2 \rightarrow n = 3 \neq m^\vee = 4$  we obtain the model denoted  $\mathcal{A}^{(3|4)}(-\frac{1}{\beta_0}, -\frac{i}{\beta_0})$ . These two models describe respectively the non-simply laced  $D_3^{(2)}$  and  $C_2^{(1)}$  ATFTs.

- $n_1^0 = 1$  : for  $m_0 = m_1 = 1$  ( $\Delta = 0$ )  $\rightarrow m = m^\vee = n = 3$  we obtain the model denoted  $\mathcal{A}^{(3|3)}(\sqrt{\frac{3}{2}}\beta_0, \frac{i}{\sqrt{2}}\beta_0)$ . It is self-dual and describes the simply laced  $A_2^{(1)}$  ATFT. These three models are all generated by a fundamental multicomplex number  $e$  satisfying

$e^3 = -1$  (or its dual  $e^\vee$ ) which possesses the  $(m \times m)$  matrix representation  $\pi'$  :

$$\mathcal{A}^{(3|m)}(\alpha_0, \beta_0) \text{ with respect to } \pi'[e] = \text{Diag}[\underbrace{e^{i\frac{\pi}{3}} \dots e^{i\frac{\pi}{3}}}_{(m_0 \text{ times})}, \underbrace{-1 \dots -1}_{(m_1 \text{ times})}, \underbrace{e^{-i\frac{\pi}{3}} \dots e^{-i\frac{\pi}{3}}}_{(m_0 \text{ times})}]$$

$\updownarrow$  dual with

$$\mathcal{A}^{(3|m^\vee)}(\alpha_0^\vee, \beta_0^\vee) \text{ with respect to } \pi'[e^\vee] = \text{Diag}[\underbrace{e^{i\frac{\pi}{3}} \dots e^{i\frac{\pi}{3}}}_{(m_0^\vee \text{ times})}, \underbrace{-1 \dots -1}_{(m_1^\vee \text{ times})}, \underbrace{e^{-i\frac{\pi}{3}} \dots e^{-i\frac{\pi}{3}}}_{(m_0^\vee \text{ times})}].$$

Consider now the multisine-Gordon model for  $n = 4$ . The  $q$ -deformed structure of the non-local charge algebra is ensured for the choices  $n_1^0 = 0$  and  $n_4 = 0$ . The resulting model  $\mathcal{A}^{(4|4)}$  corresponds to the simply laced  $A_3^{(1)}$  ATFT. Since its hidden symmetry is self-dual under transformations (2.9) and (2.10), the weak-strong coupling regimes (with respect to the parameter  $\beta_0$ ) of the two dual actions are identical [2]. However, while the fundamental multicomplex number representation associated with one model is :

$$\begin{aligned} \mathcal{A}^{(4|m)}(\alpha_0, \alpha_1; \beta_0) \text{ generated by :} \\ \pi'[e] = \text{Diag}[\underbrace{e^{i\frac{\pi}{4}} \dots e^{i\frac{\pi}{4}}}_{(m_0 \text{ times})}, \underbrace{e^{i\frac{3\pi}{4}} \dots e^{i\frac{3\pi}{4}}}_{(m_1 \text{ times})}, \underbrace{e^{-i\frac{3\pi}{4}} \dots e^{-i\frac{3\pi}{4}}}_{(m_1 \text{ times})}, \underbrace{e^{-i\frac{\pi}{4}} \dots e^{-i\frac{\pi}{4}}}_{(m_0 \text{ times})}], \end{aligned}$$

its dual representation, associated with the dual model is obtained from  $\frac{m_0^\vee}{m_1^\vee} = \frac{m_1}{m_0}$ , which corresponds to :

$$\begin{aligned} \mathcal{A}^{(4|m^\vee)}(\alpha_0^\vee, \alpha_1^\vee; \beta_0^\vee) \text{ generated by :} \\ \pi'[e^\vee] = \text{Diag}[\underbrace{e^{i\frac{\pi}{4}} \dots e^{i\frac{\pi}{4}}}_{\left(\begin{smallmatrix} m_0^\vee = m_1 \\ \text{times} \end{smallmatrix}\right)}, \underbrace{e^{i\frac{3\pi}{4}} \dots e^{i\frac{3\pi}{4}}}_{\left(\begin{smallmatrix} m_1^\vee = m_0 \\ \text{times} \end{smallmatrix}\right)}, \underbrace{e^{-i\frac{3\pi}{4}} \dots e^{-i\frac{3\pi}{4}}}_{\left(\begin{smallmatrix} m_1^\vee = m_0 \\ \text{times} \end{smallmatrix}\right)}, \underbrace{e^{-i\frac{\pi}{4}} \dots e^{-i\frac{\pi}{4}}}_{\left(\begin{smallmatrix} m_0^\vee = m_1 \\ \text{times} \end{smallmatrix}\right)}]. \end{aligned}$$

Then, self-duality of  $A_3^{(1)}$  translates into the following exchange :

$$e \longleftrightarrow e^\vee = -\frac{1}{e} \quad (4.49)$$

in the multicomplex space.

Action (4.40) in the multicomplex space (and its dual multicomplex) and action (4.43) in the usual complex space provide a unified representation of all lowest-rank affine Toda field theories. If the perturbation is relevant, no new operator will be generated under renormalization flows. Non-local MC-currents are conserved to all order in CPT and consequently quantum duality is satisfied. Tables 1.1-1.3 give the list of ATFTs described in this formalism whereas the exchange of the node under multicomplex conjugation in the corresponding Dynkin diagrams is depicted in fig. 1.1-1.3.

However, if the perturbation becomes marginal, MC-vertex operators associated to non-simple roots are generated under renormalization. For instance, let us consider the simply-laced cases ( $A_2^{(1)}$ ,  $A_3^{(1)}$  and  $D_4^{(1)}$ ) in Table 1.1. The perturbation is marginal if  $\beta_0^2 = 1$ . The perturbing (self-dual) operator in MC-space can be written in the standard complex space as a current-current perturbation of a level-one WZNW model. We obtain respectively  $su(3)$ ,  $su(4)$  and  $so(8)$  current-current perturbations<sup>6</sup>[9]. This suggests a possible representation of WZNW or Gross-Neveu models in MC-space.

## 5 MC-algebra and perturbed conformal field theories

Instead of considering action (2.1) (or its dual counterpart to first order in CPT) as a perturbed  $(n - 1)$  free field CFT, we can proceed differently. Let us consider the MSG potential  $\sum_l x^{(l)}$  for  $l \in \{0, \dots, n-1\}$ . If we truncate this potential by supressing one of the MC-operators, say  $x^{(0)}$ , the resulting form is no longer invariant under multicomplex conjugation. However, conformal invariance can be realized by adding some specific MC-charge at infinity coupled to the fundamental field of the theory.  $x^{(0)}$  is then identified as the perturbation in MC-space. To show that, we define the holomorphic part of the MC-stress-energy tensor :

$$T(z) = -\frac{1}{2}(\partial\Phi)^2 + \sqrt{2}\beta_0\mathbf{Q}.\partial^2\Phi, \quad (5.50)$$

where  $\mathbf{Q} \in \mathbb{MC}_{(n|m)}$  and similarly for the antiholomorphic part. For further convenience<sup>7</sup>, we write  $\mathbf{Q}$  as :

$$\mathbf{Q} = \sum_{b=0}^{n/2-1} \mathbf{Q}_b(-P_b^2). \quad (5.51)$$

For  $n$  even, using the representation (4.47) we define the  $n/2$  projections  $\pi_a$  over the usual complex space with :

$$\pi_a(x^{(0)}) = \left(\pi[x^{(0)}]\right)_{aa} = e^{\beta_0\mathbf{r}_a\cdot\Phi} \quad \text{for } a \in \{0, \dots, n/2 - 1\}. \quad (5.52)$$

From eq. (5.52), we see that although the MSG Lagrangian representation does *not* depend on the choice of a particular projection, the Lagrangian representation in the usual complex space of the CFT part (i.e. equivalently the expression of the perturbation in the complex space) depends on this choice. In fact, due to the permutation symmetry of the “parametrized” roots  $\mathbf{r}_a$  for  $a \in \{0, \dots, n/2 - 2\}$  (see Appendix B) the differences between all possible perturbations reduce to two distinct cases. In the first case, we

---

<sup>6</sup>The resulting action is of the Gross-Neveu type, obtained from the bosonization of the  $\mathcal{G}$ -invariant Gross-Neveu models.

<sup>7</sup>It is also possible to expand  $\mathbf{Q}$  over generators  $P_b$  instead of  $-P_b^2$ . In any case, the central charge takes real values.

use the projection<sup>8</sup>  $\pi_{n/2-1}$  and the corresponding QFT is denoted  $\mathcal{P}$ . In the second case, we proceed similarly and use the projection  $\pi_0$ . The corresponding QFT is then denoted  $\overline{\mathcal{P}}$ . Each charge  $\mathbf{Q}_b$  is then associated to the CFT obtained from the projection  $\pi_b$ . Consequently, an appropriate choice of the charge  $\mathbf{Q}$ , e.g. of the set  $\{\mathbf{Q}_b\}$  ensures *simultaneously* the conformal invariance of *all* CFTs. From these previous remarks, it reduces to study only  $\mathcal{P}$  and  $\overline{\mathcal{P}}$ , i.e. to calculate  $\mathbf{Q}_{n/2-1}$  and  $\mathbf{Q}_0$ . It translates into a condition on the conformal dimensions  $\Delta_{n/2-1}$  and  $\Delta_0$  of the vertex operators :

$$\begin{aligned} \Delta_b \left( e^{\beta_0 \mathbf{r}_a \cdot \Phi} \right) = 1 \quad & \text{for } \mathcal{P} (\pi_{n/2-1}), \text{ i.e. for all } a \in \{0, \dots, \frac{n}{2} - 2, \frac{n}{2}, \dots, n-2\}, \\ \text{or} \quad & \text{for } \overline{\mathcal{P}} (\pi_0), \text{ i.e. for all } a \in \{1, \dots, n-1\}. \end{aligned} \quad (5.53)$$

Using eq. (5.50) and (5.51), the holomorphic conformal dimension of each vertex operator is :

$$\Delta_b \left( e^{\beta_0 \mathbf{r}_a \cdot \Phi} \right) = -\frac{\beta_0^2}{2} \mathbf{r}_a^2 + \sqrt{2} \beta_0 \mathbf{Q}_b \cdot \mathbf{r}_a. \quad (5.54)$$

For further convenience and by analogy with ATFTs approach, let us introduce :

$$\mathbf{Q}_b = \frac{1}{\sqrt{2}} [\beta_0 \rho_b + \beta_0^\vee \rho_b^\vee] \quad \text{where} \quad \rho_b = \sum_{\{c\}} \omega_{c;b} \quad \text{and} \quad \rho_b^\vee = \sum_{\{c\}} \omega_{c;b}^\vee \quad (5.55)$$

with  $c \neq n/2 - 1$  for  $\mathcal{P}$  and  $c \neq 0$  for  $\overline{\mathcal{P}}$ . Eqs. (5.53) can be satisfied if  $\omega_{c;b}, \omega_{c;b}^\vee$  are choosed to obey :

$$\omega_{c;b}^\vee \cdot \mathbf{r}_a = \delta_{ac} \frac{1}{\beta_0 \beta_0^\vee}. \quad (5.56)$$

Similarly, the same approach can be applied to the  $n$  odd case and the corresponding results are obtained using the substitutions (3.26). In each case ( $\mathcal{P}$  or  $\overline{\mathcal{P}}$ ) and any value of  $n$ , it is straightforward to compute the MC-central charge of the conformally invariant part:

$$c = n - 1 + 24|\mathbf{Q}|^2. \quad (5.57)$$

We notice that expression (5.55) is self-dual under the duality transformation (2.9). From the above analysis, we have computed the “parametrized” central charges  $c_b = \pi_b(c)$  for  $\mathcal{P}$  and  $\overline{\mathcal{P}}$ , expressed in terms of  $n$ ,  $(m_a, m_a^\vee)$ ,  $(m, m^\vee)$ ,  $\{\mathbf{r}_a\}$  and  $\{\beta_a\}$ . The “parametrized” co-weights are given in Appendix B and their associated weights are obtained using transformations (2.9). Using eq. (5.57) with eqs. (5.51) and (5.55) we obtain :

---

<sup>8</sup>In fact for  $n$  even (and similarly for  $n$  odd) all projections  $\pi_a$  for  $a \in \{0, \dots, n/2 - 2\}$  are equivalent under multicomplex conjugation up to a permutation of the roots. For  $n$  odd, using substitutions (3.26), it is obvious to see that the nodes associated respectively to  $n/2 - 1$  and  $n - 1$  “collapse” together as  $\alpha_{n/2-1} \rightarrow 0$ .

- For  $\mathcal{P}$  :

$$\begin{aligned} c_{n/2-1} &= n - 1 + 12 \left[ \Gamma_{n;n/2-1}^{(1)} + \Gamma_{n;n/2-1}^{(2)} \right] && \text{for } n \text{ even,} \\ c_{(n-1)/2} &= n - 1 + 12 \left[ \Gamma_{n;(n-1)/2}^{(1)} \right] && \text{for } n \text{ odd.} \end{aligned} \quad (5.58)$$

- For  $\overline{\mathcal{P}}$  and  $n$  even :

$$c_0 = n - 1 + 12 \left[ \Gamma_{n;0}^{(1)} + \Gamma_{n;0}^{(2)} + \Gamma_{n;0}^{(3)} \right] \quad (5.59)$$

and where we define for  $n$  even :

$$\begin{aligned} \Gamma_{n;b}^{(1)} &= \sum_{\substack{a=0 \\ a \neq b}}^{n/2-2} \left[ \left( \frac{\mathbf{r}_a^2}{2} \beta_0 + \frac{1}{\beta_0} \right)^2 \left( \frac{\beta_0}{\beta_a} \right)^2 \right], \\ \Gamma_{n;b}^{(2)} &= - \left( \frac{\sum_{a=0}^{n/2-1} m_a \mathbf{r}_a^2}{2m_b} \beta_0 + \frac{m}{2m_b} \frac{1}{\beta_0} \right)^2 \left( \frac{\beta_0}{\alpha_b} \right)^2, \\ \Gamma_{n;0}^{(3)} &= \left( \frac{\sum_{a=1}^{n/2-1} m_a \mathbf{r}_a^2}{2m_0} \beta_0 + \frac{m - 2m_0}{2m_0} \frac{1}{\beta_0} \right)^2. \end{aligned} \quad (5.60)$$

For  $\overline{\mathcal{P}}$  and  $n$  odd, we use substitution (3.26).

The conformal dimension of each vertex operator, except the perturbation, is one. Then, their integrals appear naturally as “screening charges” whereas the conformal dimension of the perturbation for any projection  $\pi_b$  is :

$$\Delta_b \left( \pi_b(x^{(0)}) \right) = \Delta_b \left( e^{\beta_0 \mathbf{r}_b \cdot \Phi} \right) = 1 - \left[ \frac{m^\vee}{m_b^\vee} \left( \beta_0^2 \frac{\mathbf{r}_b^2}{2} \right) + \frac{m}{m_b} \right]. \quad (5.61)$$

For  $(\{\alpha_a\} \in \mathbb{R}, \{\beta_a\} \in i\mathbb{R})$ , the perturbation is relevant ( $\Delta_b < 1$ ) or marginal ( $\Delta_b = 1$ ) iff:

$$(\beta_0^\mathbb{R})^2 \frac{\mathbf{r}_b^2}{2} \leq \frac{\frac{m}{m_b}}{\frac{m^\vee}{m_b^\vee}}. \quad (5.62)$$

The renormalizability property of the model is then encoded in the MC-algebra.

As we saw previously, depending on the dimension, the specific structure of the MC-algebra (e.g. the values of  $m_a$ ) and ratios of the parameters, the resulting model possesses a  $q$ -deformed symmetry. For each projection which leads to  $\mathcal{P}$  or  $\overline{\mathcal{P}}$ , we have computed for the ratios reported in Table 1.1 the central charge associated with the truncated MSG model, i.e. without the perturbing term  $x^{(0)}$ . For instance, for projection  $\mathcal{P}$  for case a/ and b/ the truncated model is identified to an  $su(2) \otimes su(2)$  theory (two decoupled Liouville theories), whereas case e/ corresponds to an  $so(4) \otimes so(4)$  theory (four decoupled Liouville theories). For projection  $\overline{\mathcal{P}}$ , Toda field theories are obtained. For any projection, case c/ and d/ are respectively identified with  $A_2$  and  $A_3$  Toda field theories. Using the results reported in Table 1.1, it can be checked that agreement is obtained with the results found

in [15]. However, for imaginary values of the parameters  $\beta_a$ , a truncation of the Hilbert space is necessary in order to obtain a unitary theory. As ratios are fixed, the values of  $\beta_0$  are fixed in each case, corresponding to a quantum group restriction of the model. This last restriction corresponds to points in parameter space.

For generic values of the parameters the situation is much more complicated. Although the central charge associated with the truncated MSG model can be computed, it is not clear if unitary<sup>9</sup> representations of the Virasoro algebra always exist [16]. However, for  $\mathbf{r}_a^2 = 2$  and  $a \in \{0, \dots, \frac{n}{2} - 1\}$  (which is the standard convention for simply laced affine Lie algebra), we notice that any “truncation” ( $\mathcal{P}$  or  $\overline{\mathcal{P}}$ ) of the basis of MC-vertex operators in action (2.1) leads to a self-dual CFT (ratios of parameters are fixed) and condition (5.62) becomes  $(\beta_0^{\mathbb{R}})^2 \leq 1$ .

## 6 Concluding remarks

Consider the MSG model with action  $\mathcal{A}^{(n|m)}(\eta)$  generated by the MC-algebra  $\mathbb{MC}_{(n|m)}$ . The parameter space can be described as follow. In the deep ultraviolet, it exists a discret set of one-dimensional submanifolds  $\mathcal{S}_0 = (\{\alpha_a^{(0)}\}, \{\beta_a^{(0)}\})$  (see Tables 1.1-1.3) which corresponds to a scale invariant theory, i.e a CFT. This CFT with MC-central charge (5.57) possess a Lagrangian representation in MC-space in terms of a truncated basis of MC-vertex operators,  $\sum_{l=1}^{n-1} x^{(l)}$  for instance. In usual complex space, it admits two unequivalent Lagrangian representations  $\mathcal{P}$  and  $\overline{\mathcal{P}}$ . A parametrized central charge  $c_b = c_b(\{\alpha_a^{(0)}\}, \{\beta_a^{(0)}\})$  for  $b = 0$  or  $E((n+1)/2) - 1$  was obtained for each representations, describing the fixed points of different known CFTs like decoupled Toda or Toda field theories.

We are now interested in the neighbourhood region of these fixed points where we loose the scale invariance of the model. As the CFT in MC-space is not invariant under multicomplex conjugation, a natural perturbation is then provided by imposing the MC-conjugation symmetry to the resulting model. Indeed this perturbation, say  $x^{(0)}$ , does not correspond to an *arbitrary* deformation of the CFT. The perturbed model always corresponds to a one-parameter family of integrable massive QFTs. For  $\mathcal{S}_0^+ = (\{\alpha_a^{(0)}\} \in \mathbb{R}, \{\beta_a^{(0)}\} \in i\mathbb{R})$ , the model is identified to lowest-rank imaginary coupling ATFTs which possess soliton solutions. However, unitarity is only assumed at specific points on  $\mathcal{S}_0^+$ . The theory gets truncated and the hidden symmetry acts as a parametrized quantum group with the deformation parameters  $q_b(\{\alpha_a^{(0)}\}, \{\beta_a^{(0)}\})$  being a root of unity. For  $\mathcal{S}_0^- = (\{\alpha_a^{(0)}\} \in i\mathbb{R}, \{\beta_a^{(0)}\} \in \mathbb{R})$ , lowest-rank real coupling ATFTs are obtained.

Exept in the Kosterlitz-Thouless region of  $\mathcal{S}_0$ , no new terms are generated under the renormalization group flow. Non-local MC-conserved currents  $\{J_{\eta^{(k)\vee}}, \overline{J}_{\eta^{(k)\vee}}\}$  exist which are conserved to *all* orders in CPT. There, the MSG model (2.11) with (2.12) admits a dual Lagrangian representation with action  $\mathcal{A}^{(n|m^\vee)}(\eta^\vee)$ , generated by the dual multicomplex algebra  $\mathbb{MC}_{(n|m^\vee)}$ . The weak coupling regime of one MSG and the strong coupling regime of its dual are simply related through the parameter exchange  $\beta_0 \leftrightarrow 1/\beta_0$ .

---

<sup>9</sup>However, many statistical systems do not satisfy the unitarity condition (nonself intersecting polymer chains, magnetics with stochastic interactions, etc...).

The neighbourhood region of  $\mathcal{S}_0$  ( $\lambda \ll 1$ ) is much more complicated but a few remarks can be done from our previous analysis. First, the non-local currents (and dual ones) are still conserved, at least to first order in CPT, for a larger parameter space  $\mathcal{S}_1$  described by eqs. (2.16), (2.17), (2.18). It is easy to see that the symmetry group of  $\mathcal{S}_1^+$  (and similarly for  $\mathcal{S}_1^-$  with the opposite signature) is the pseudo-rotational group  $SO(\frac{n}{2} - R, \frac{n}{2} - 1)$  for  $n$  even and  $SO(\frac{n-1}{2} - R, \frac{n-1}{2})$  for  $n$  odd. However, the conserved charges (2.19) do not satisfy the previous  $q$ -deformed symmetry since braiding relations between MC-operators strongly depend on the ratios of parameters. A less restrictive algebraic structure, like more general Hopf algebras, may exist. It is known that integrable models can be generated by some more general algebras than the quantum Lie algebras : the integrability condition is dictated by the Yang-Baxter equation itself [17]. Secondly, the non-local conserved currents does *not* necessary form a closed algebra. Then, renormalization group flow may generated counterterms which can spoiled the current conservation to higher order in CPT.

To conclude, we would like to mention that *all* the integrable (dual-)QFTs in [3, 4] and here are generated by the MC-algebra  $\mathbb{MC}_{(n|m)}$  for specific values of  $(n, m)$ . The difference between them simply reduces to the parameter space. Particular, the non-local conserved charges (2.19), if they satisfy a  $q$ -deformed algebraic structure [7], can be simply expressed in terms of a Lie algebra based on the field  $\mathbb{MC}_{(n|m)}$  (multicomplex are of characteristic 0 as for the usual complex numbers), where parameters  $(\alpha_a, \beta_a)$  play the role of deformations. Furthermore, it is believed that more general MC-algebras exist [4]. Many other QFTs should then be described within this formalism.

**Acknowledgements** We are very grateful to C. Ahn, J.L. Kneur, A. Neveu, M. Rausch de Traubenberg, F. Smirnov, G. Takács and J. Thierry-Mieg for useful discussions. P.B. thanks for the hospitality of KIAS where part of this work was done. Work supported in part by the EU under contract ERBFMRX CT960012.

## Appendix A

	$n$	$R$	a	$n_1^a$	$n_2^a$	$n_3^a$	$A_a/B_0$	$A_{\frac{n}{2}-1}/B_0$	$B_a/B_0$	$\delta_a$	Algebra
a/	4	1	0	*	2	1	0	-1	1	1	$(n_4 = 0) C_2^{(1)}$
b/	4	1	0	*	1	2	0	-1/4	1	1/2	$(n_4 = 0) D_3^{(2)}$
c/	4	1	0	*	1	1	0	-3/4	1	1/2	$(n_4 = 1) A_2^{(1)}$
d/	4	2	0	*	4	1	0	0	1	2	$A_2^{(2)}$
e/	4	2	0	*	1	4	0	0	1	1/2	$A_2^{(2)}$
f/	4	2	0	*	2	2	0	0	1	1	$A_1^{(1)}$
g/	6	2	0 1	* 0	1 1	2 1	0 -1/4	0	1 1/4	1/2 1	$A_5^{(2)}$
h/	6	2	0 1	* 0	2 1	1 1	0 -1	0	1 1	1 1	$B_3^{(1)}$
i/	6	3	0 1	* *	1 1	1 3	0 0	0	1 3	1/2 1/2	$D_4^{(3)}$
j/	6	3	0 1	* *	1 3	1 1	0 0	0	1 1/3	1/2 3/2	$G_2^{(1)}$
k/	6	3	0 1	* *	2 2	1 1	0 0	0	1 1	1 1	$D_3^{(2)}$
l/	6	3	0 1	* *	1 2	2 1	0 0	0	1 1/4	1/2 1	$A_4^{(2)}$
m/	6	3	0 1	* *	1 1	2 2	0 0	0	1 1	1/2 1/2	$C_2^{(1)}$
n/	6	2	0 1	* *	1 1	1 1	0 0	-1/2	1 1	1/2 1/2	$(n_4 = 0) A_3^{(1)}$
o/	8	3	0 1 2	* * 0	1 1 1	1 1 1	0 0 -1/2	0	1 1 1/2	1/2 1/2 1	$D_4^{(1)}$
p/	8	4	0 1 2	* * *	1 1 1	2 1 1	0 0 0	0	1 1/2 1/2	1/2 1/2 1/2	$A_5^{(2)}$
q/	8	4	0 1 2	* * *	2 1 1	1 1 1	0 0 0	0	1 2 2	1 1/2 1/2	$B_3^{(1)}$
r/	10	5	0 1 2 3	* * * *	1 1 1 1	1 1 1 1	0 0 0 0	0	1 1 1 1	1/2 1/2 1/2 1/2	$D_4^{(1)}$

Table 1.2 - Solutions for the even case with restrictions, and the corresponding algebras of rank  $r = n - 1 - R$ .  $R$  : number of restrictions ( $R = s + 1$  if  $A_{n/2-1} \neq 0$ , else  $R = s + 2$ ).  
 \* stands for one of the  $s + 1$  restrictions as defined in (3.33)

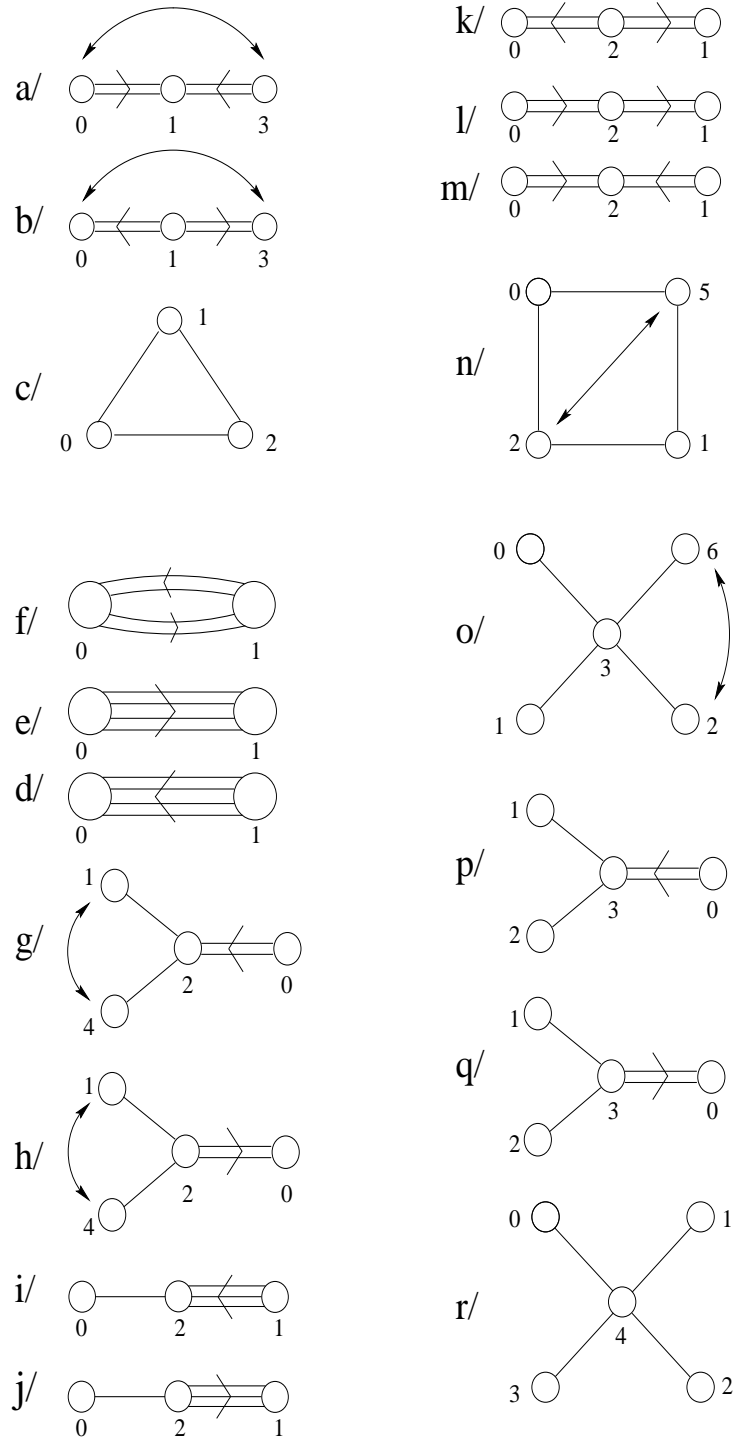


Figure 1.2 - Dynkin diagrams corresponding to affine Lie algebras Table 1.2 .

	$n$	$R$	$a$	$n_1^a$	$n_2^a$	$n_3^a$	$A_a/B_0$	$B_a/B_0$	$\delta_a$	Algebra
a/	3	1	0	*	4	1	0	1	1	$A_2^{(2)}$
b/	3	1	0	*	1	4	0	1	1/4	$A_2^{(2)}$
c/	3	1	0	*	2	2	0	1	1/2	$A_1^{(1)}$
d/	5	1	0	*	1	2	0	1	1/4	$A_5^{(2)}$
			1	0	1	1	-1/4	1/4	1/2	
e/	5	1	0	*	2	1	0	1	1/2	$B_3^{(1)}$
			1	0	1	1	-1	1	1/2	
f/	5	2	0	*	1	1	0	1	1/4	$D_4^{(3)}$
			1	*	1	3	0	3	1/4	
g/	5	2	0	*	1	1	0	1	1/4	$G_2^{(1)}$
			1	*	3	1	0	1/3	3/4	
h/	5	2	0	*	1	2	0	1	1/4	$C_2^{(1)}$
			1	*	1	2	0	1	1/4	
i/	5	2	0	*	1	2	0	1	1/4	$A_4^{(2)}$
			1	*	2	1	0	1/4	1/2	
j/	5	2	0	*	2	1	0	1	1/2	$D_3^{(2)}$
			1	*	2	1	0	1	1/2	
k/	7	2	0	*	1	1	0	1	1/4	$D_4^{(1)}$
			1	*	1	1	0	1	1/4	
			2	0	1	1	-1/2	1/2	1/2	
l/	7	3	0	*	1	2	0	1	1/4	$A_5^{(2)}$
			1	*	1	1	0	1/2	1/4	
			2	*	1	1	0	1/2	1/4	
m/	7	3	0	*	2	1	0	1	1/2	$B_3^{(1)}$
			1	*	1	1	0	2	1/4	
			2	*	1	1	0	2	1/4	
n/	9	4	0	*	1	1	0	1	1/4	$D_4^{(1)}$
			1	*	1	1	0	1	1/4	
			2	*	1	1	0	1	1/4	
			3	*	1	1	0	1	1/4	

Table 1.3 - Solutions for the odd case with restrictions, and the corresponding algebras.  
of rank  $r = n - 1 - R$ .  $R$  : number of restrictions :  $R = s + 1$ .  
\* stands for one of the  $s + 1$  restrictions as defined in (3.36).

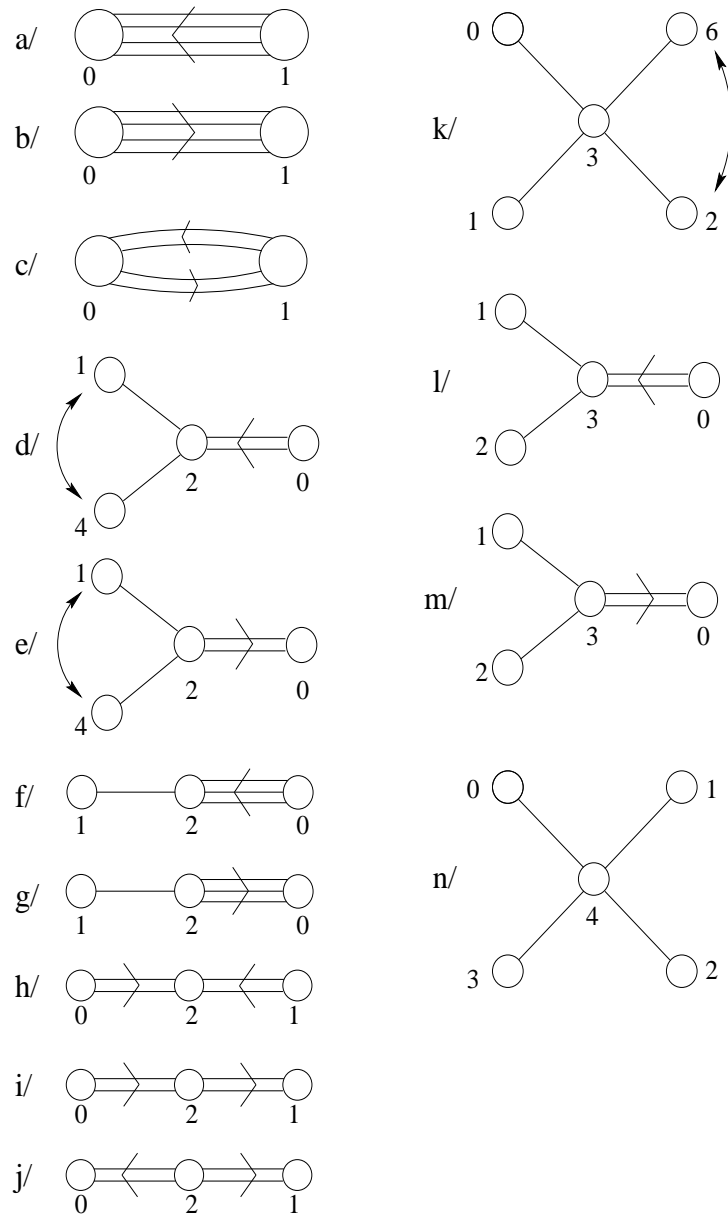


Figure 1.3 - Dynkin diagrams corresponding to affine Lie algebras Table 1.3 .

## Appendix B

### • $n$ even.

The set of “parametrized” roots for  $n$  even reads, with  $a \in \{0, \dots, \frac{n}{2} - 2\}$  :

$$\begin{aligned} \mathbf{r}_a &= \left[0, \dots, 0, i\frac{\alpha_a}{\beta_0}, \frac{\beta_a}{\beta_0}, 0, \dots, 0\right], & \mathbf{r}_{a+\frac{n}{2}} &= \left[0, \dots, 0, -i\frac{\alpha_a}{\beta_0}, \frac{\beta_a}{\beta_0}, 0, \dots, 0\right], \\ \mathbf{r}_{\frac{n}{2}-1} &= \left[0, -\frac{m_0}{m_{\frac{n}{2}-1}}, \dots, 0, -\frac{\beta_a m_a}{\beta_0 m_{\frac{n}{2}-1}}, \dots, 0, -\frac{\beta_{\frac{n}{2}-2} m_{\frac{n}{2}-2}}{\beta_0 m_{\frac{n}{2}-1}}, i\frac{\alpha_{\frac{n}{2}-1}}{\beta_0}\right], \\ \mathbf{r}_{n-1} &= \left[0, -\frac{m_0}{m_{\frac{n}{2}-1}}, \dots, 0, -\frac{\beta_a m_a}{\beta_0 m_{\frac{n}{2}-1}}, \dots, 0, -\frac{\beta_{\frac{n}{2}-2} m_{\frac{n}{2}-2}}{\beta_0 m_{\frac{n}{2}-1}}, -i\frac{\alpha_{\frac{n}{2}-1}}{\beta_0}\right]. \end{aligned} \quad (6.63)$$

For  $\mathcal{P}$ , the set of dual “parametrized” co-weights defined by (5.56) reads, with  $a \in \{0, \dots, \frac{n}{2} - 2\}$  :

$$\begin{aligned} \beta_0^\vee \omega_{a; \frac{n}{2}-1}^\vee &= \left[0, \dots, 0, \frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0, -\frac{m_a}{2im_{\frac{n}{2}-1}\alpha_{\frac{n}{2}-1}}\right], \\ \beta_0^\vee \omega_{a+\frac{n}{2}; \frac{n}{2}-1}^\vee &= \left[0, \dots, 0, -\frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0, -\frac{m_a}{2im_{\frac{n}{2}-1}\alpha_{\frac{n}{2}-1}}\right], \\ \beta_0^\vee \omega_{n-1; \frac{n}{2}-1}^\vee &= \left[0, \dots, 0, -\frac{1}{i\alpha_{\frac{n}{2}-1}}\right]. \end{aligned} \quad (6.64)$$

For  $\overline{\mathcal{P}}$ , the set of dual “parametrized” co-weights reads, with  $a \in \{1, \dots, \frac{n}{2} - 2\}$  is :

$$\begin{aligned} \beta_0^\vee \omega_{a;0}^\vee &= \left[-\frac{m_a}{2im_0\alpha_0}, -\frac{m_a}{2m_0\beta_0}, 0, \dots, 0, \frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0\right], \\ \beta_0^\vee \omega_{a+\frac{n}{2};0}^\vee &= \left[-\frac{m_a}{2im_0\alpha_0}, -\frac{m_a}{2m_0\beta_0}, 0, \dots, 0, -\frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0\right], \\ \beta_0^\vee \omega_{\frac{n}{2}-1;0}^\vee &= \left[-\frac{m_{\frac{n}{2}-1}}{2im_0\alpha_0}, -\frac{m_{\frac{n}{2}-1}}{2m_0\beta_0}, 0, \dots, 0, \frac{1}{2i\alpha_{\frac{n}{2}-1}}\right], \\ \beta_0^\vee \omega_{n-1;0}^\vee &= \left[-\frac{m_{\frac{n}{2}-1}}{2im_0\alpha_0}, -\frac{m_{\frac{n}{2}-1}}{2m_0\beta_0}, 0, \dots, 0, -\frac{1}{2i\alpha_{\frac{n}{2}-1}}\right], \\ \beta_0^\vee \omega_{\frac{n}{2};0}^\vee &= \left[-\frac{1}{i\alpha_0}, 0, \dots, 0\right]. \end{aligned} \quad (6.65)$$

Similarly,  $\omega_a$  are obtained using substitutions  $\alpha_a \rightarrow \alpha_a^\vee$ ,  $\beta_a \rightarrow \beta_a^\vee$  and  $m_a \rightarrow m_a^\vee$ .

### • $n$ odd.

For  $n$  odd the set of “parametrized” roots is given by ( with  $a \in \{0, \dots, \frac{n-1}{2} - 1\}$  ) :

$$\begin{aligned} \mathbf{r}_a &= \left[0, \dots, 0, i\frac{\alpha_a}{\beta_0}, \frac{\beta_a}{\beta_0}, 0, \dots, 0\right], & \mathbf{r}_{a+\frac{n+1}{2}} &= \left[0, \dots, 0, -i\frac{\alpha_a}{\beta_0}, \frac{\beta_a}{\beta_0}, 0, \dots, 0\right], \\ \mathbf{r}_{\frac{n-1}{2}} &= \left[0, -\frac{2m_0}{m_{\frac{n}{2}-1}}, \dots, 0, -\frac{2\beta_a m_a}{\beta_0 m_{\frac{n}{2}-1}}, \dots, 0, -\frac{2\beta_{\frac{n}{2}-2} m_{\frac{n}{2}-2}}{\beta_0 m_{\frac{n}{2}-1}}\right]. \end{aligned} \quad (6.66)$$

For  $\mathcal{P}$ , the set of dual “parametrized” co-weights is then, with  $a \in \{0, \dots, \frac{n-1}{2} - 1\}$  :

$$\begin{aligned}\beta_0^\vee \omega_{a; \frac{n-1}{2}}^\vee &= [0, \dots, 0, \frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0], \\ \beta_0^\vee \omega_{a+\frac{n+1}{2}; \frac{n-1}{2}}^\vee &= [0, \dots, 0, -\frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0].\end{aligned}\tag{6.67}$$

For  $\overline{\mathcal{P}}$ , the set of dual “parametrized” co-weights is, with  $a \in \{1, \dots, \frac{n-1}{2} - 1\}$  :

$$\begin{aligned}\beta_0^\vee \omega_{a;0}^\vee &= [-\frac{m_a}{2im_0\alpha_0}, -\frac{m_a}{2m_0\beta_0}, 0, \dots, 0, \frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0], \\ \beta_0^\vee \omega_{a+\frac{n+1}{2};0}^\vee &= [-\frac{m_a}{2im_0\alpha_0}, -\frac{m_a}{2m_0\beta_0}, 0, \dots, 0, -\frac{1}{2i\alpha_a}, \frac{1}{2\beta_a}, 0, \dots, 0], \\ \beta_0^\vee \omega_{\frac{n-1}{2};0}^\vee &= [-\frac{m_{\frac{n-1}{2}}}{2im_0\alpha_0}, -\frac{m_{\frac{n-1}{2}}}{2m_0\beta_0}, 0, \dots, 0], \\ \beta_0^\vee \omega_{\frac{n-1}{2}+1;0}^\vee &= [-\frac{1}{i\alpha_0}, 0, \dots, 0].\end{aligned}\tag{6.68}$$

## References

- [1] S. Coleman, Phys. Rev. **D 11** 2088 (1975);  
S. Mandelstam, Phys. Rev. **D 11** 3026 (1975).
- [2] A.E. Arinshtein, V.A. Fateev, A.B. Zamolodchikov, Phys. Lett. **B 87** (1979) 389;  
H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki, Nucl. Phys. **B 338** (1990) 689;  
G.W. Delius, M.T. Grisaru, D. Zanon, Nucl. Phys. **B 359** (1991) 125; Nucl. Phys. **B 382** (1992) 365; Nucl. Phys. **B 385** (1992) 307.
- [3] P. Baseilhac, P. Grangé and M. Rausch de Traubenberg, Mod. Phys. Lett. **A 13** (1998) 2531.
- [4] P. Baseilhac, S. Galice, P. Grangé and M. Rausch de Traubenberg, submitted to Phys.Lett. **B**.
- [5] N. Fleury, M. Rausch de Traubenberg, J.Math.Phys. **33** (1992) 3356;  
N. Fleury and M. Rausch de Traubenberg, Adv. Appl. Cliff. Alg. **4** (1994) 123.
- [6] V.A. Fateev, Phys. Lett. **B 357** (1995) 397;  
V.A. Fateev, Nucl. Phys. **B 473** (1996) 509.
- [7] P. Baseilhac, Int. Journ. Mod. Phys. **A 14** (1999) 421.
- [8] H. J. DeVega, H. Eichenherr and J. M. Maillet, Nucl. phys. **B 240** (1984) 377;  
H. J. DeVega, H. Eichenherr and J. M. Maillet, Commun. Math. phys. **92** (1984) 507.

- [9] D. Bernard and A. LeClair, Comm. Math. Phys. **142** (1991) 99;  
H. J. DeVega and V. A. Fateev, J. Phys. **A 25** (1992) 2693;  
G. W. Delius, Nucl. Phys. **B 451** (1995) 445.
- [10] N. Fleury, M. Rausch de Traubenberg and R. Yamaleev, J. Math. Anal. Appl **180**  
(1993) 431.
- [11] N. Fleury, M. Rausch de Traubenberg and R. Yamaleev, J. Math. Anal. Appl **191**  
(1995) 118.
- [12] A. B. Zamolodchikov, Int. Journ. Mod. Phys. **A 4** (1989) 4235; in Adv. Studies in  
Pure Math. vol. **19** (1989) 641.
- [13] M. T. Grisaru, A. Lerda, S. Penati, D. Zanon, Nucl. Phys. **B 346** (1990);  
D. Boyanovsky, R. Holman, Nucl. Phys. **B 358** (1991) 619.
- [14] M. T. Grisaru, S. Penati, Nucl. Phys. **B 348** (1991).
- [15] T. Hollowood and P. Mansfield, Phys. Lett. **B 226** (1989) 73.
- [16] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575;  
D. Friedan, Z. Qiu and S. Shenker, Commun. Math. Phys. **107** (1986) 535;  
P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. **103** (1986) 105.
- [17] A. Kundu and B. Basumallick, Mod. Phys. Lett. **A 7** (1992) 61;  
A. Kundu and B. Basumallick, J. Math. Phys. **A 34** (1993) 1252;  
A. Kundu, hep-th/9909042.